WEIBULL IN THE WEST END

BY

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Introduction

In his paper (1) Captain McClune says '. . . it is believed that Weibull plotting should become the primary method of naval failure and repair time data analysis'. With the aim of encouraging its greater use Weibull plotting is taught to the Engineering Management Courses and Data Analysis Courses held at the Royal Naval Engineering College, Manadon. The College therefore needs sets of accurate and interesting real life data which can be used to demonstrate the power and versatility of Weibull plotting techniques.

Reliability is defined as 'the probability that an item will perform its required function for a specified period under stated conditions'. During a search for times-to-failure information for real items, happy inspiration suggested West End shows. Their function is to entertain. The daily takings are a measure of success in this function and, in general, when a show ceases to yield an adequate profit it is taken off, i.e. it fails. A West End Show is therefore closely analogous to an equipment which fails when its performance falls below some specified level. Can the distribution of numbers of consecutive performances of West End Shows be reasonably described by a Weibull distribution and what are its characteristics? The investigation proved far more interesting and useful for instruction than was expected at the outset and is also of direct interest to 'Angels'.

Source of Data

The source of data was *Who's Who in the Theatre*. This lists all shows in London theatres which survived 250 performances or more from 1940 to 1970, giving the name, date of opening, theatre and number of performances of each. To reduce the amount of data to be analysed and to remove any variability introduced by the Country being at war and in a period of austerity during the 1940's the analysis was carried out on shows between 1950 and 1970. This gave a sample size of 308 of which 10 were still running on 31st December 1970 including, of course, the evergreen *Mousetrap* which by then had clocked up 7519 performances.

Quality of Data

It would be hard to find a better set of data for analysis. It is ample, complete, undoubtedly accurate and includes 'survivors'. Also, to a much greater extent than is usual in engineering events, it can with confidence be assumed to be independent.

The Weibull Equation

As used in reliability analysis the Weibull equation is:

$$R(t) = \exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]$$

where: R(t) is Reliability

t is the age at failure, the random variable.

γ is a location parameter defining the origin of the distribution, known also as the Minimum Life.

η is the Characteristic Life, i.e. the time by which 63·2% of the population have failed.

β is the Shape Parameter controlling the shape of the distribution.

It is helpful to relate β to the 'bathtub' curve. During early life when the hazard rate is falling β is less than one. During the middle period where the hazard rate is constant $\beta=1$. During the wear-out phase where the hazard rate is increasing β is greater than 1.

Weibull plotting paper is so constructed that a Weibull distribution whose origin has been correctly specified appears as a straight line whose slope is determined by β and whose position is determined by η.

In this example the 'number of consecutive performances' as opposed to 'time' is the random variable and so the symbol p has been used in place of t.

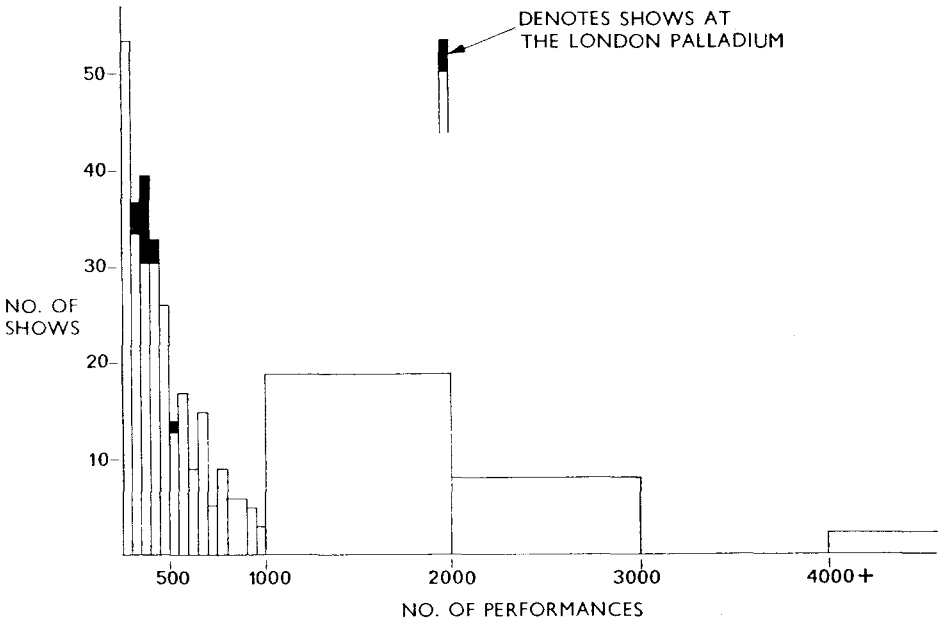

FIG. 1—HISTOGRAM

Method of Analysis

Firstly, a histogram was produced showing the number of shows in each band of 50 performances up to 1000 (FIG. 1). This immediately suggested an exponential or hyper-exponential distribution but indicated a disturbance to this trend between 300 and 450. A quick survey of this region showed the cause and shows at the London Palladium were withdrawn from the set, leaving a sample size of 293 including the 10 survivors.

This set was then ranked, median rank values calculated, as described in References 1 and 2, and plotted on Weibull paper, giving the plots shown in FIGS. 2 and 3. It is not possible to show all 283 'failures' on these plots.

250—1000 Performances

Over this range all points lie on or very close to a line with the following parameters:

Shape factor $\beta=0·86$
Characteristic life $\eta=300$
Minimum life (by definition)$=250$

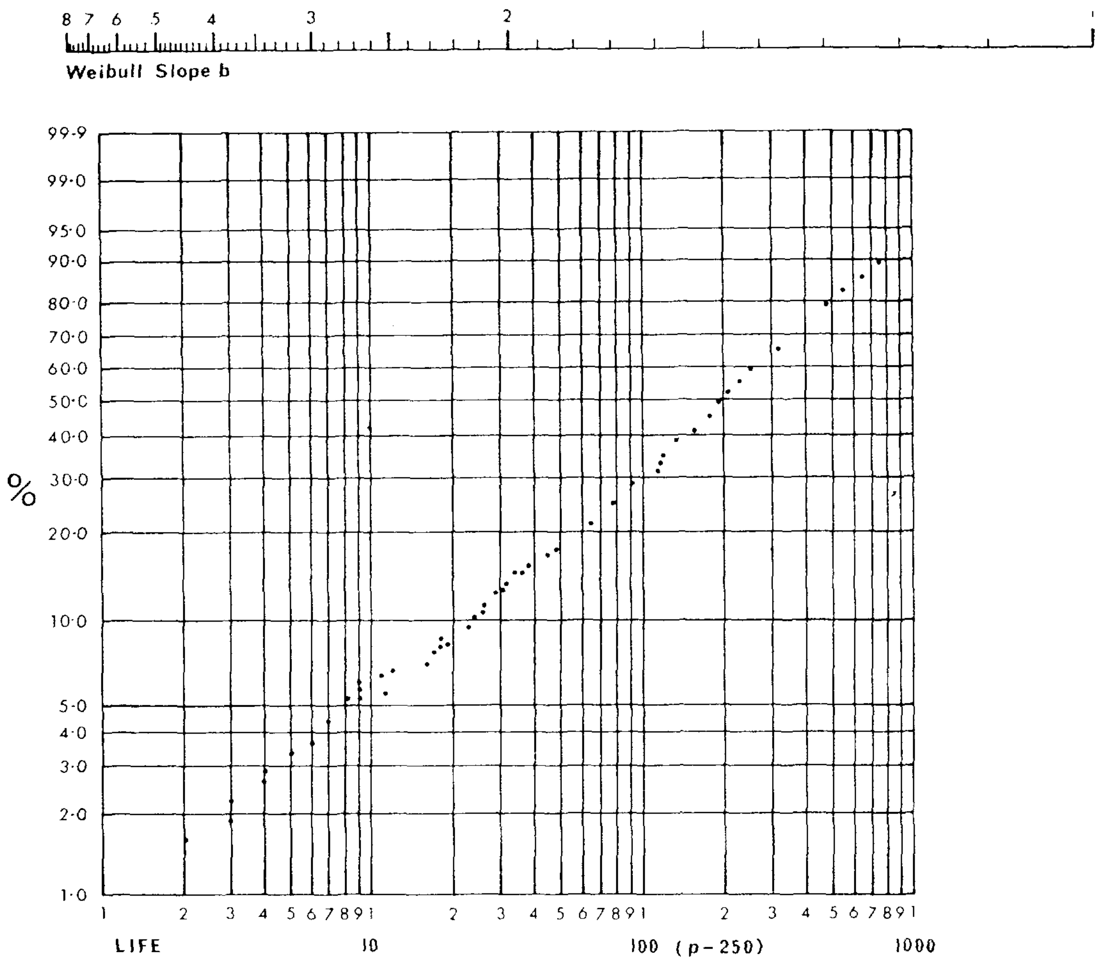

FIG. 2—WEIBULL PLOT OF LONDON SHOWS (1950-1970)—
NUMBER OF PERFORMANCES 250 TO 1250, GIVEN SURVIVAL TO 250

yielding the equation:
The probability of a show surviving p performances, given survival to 250 performances,

$$P\,(p|250) = \exp\left[-\left(\frac{p-250}{300}\right)^{.86}\right]$$

1000 Performances onwards

FIG. 3 indicates clearly a change of slope at 'age' 750. The minimum life for the whole data set is 250 so in real terms the nature of the failure time distribution changes at 750+250=1000 performances. To discover the characteristics of the latter distribution, shows which survived more than 1000 performances were re-ranked, median rank values for this sub-set were calculated and the sub-set plotted giving FIG. 4. Included in this sub-set were all surviving shows at 31st December 1970 since in theory they could have survived more than 1000 performances. They were listed and treated as 'survivors' in the sub-set.

There are far fewer points over the range 1000 performances onwards than in the range 250 to 1000 and there is greater variability either side of a best fit (by eye) straight line. However, the plot yields the following parameters:

Shape factor $\beta=1$
Characteristic life $\eta=850$
Minimum life (by definition)=1000

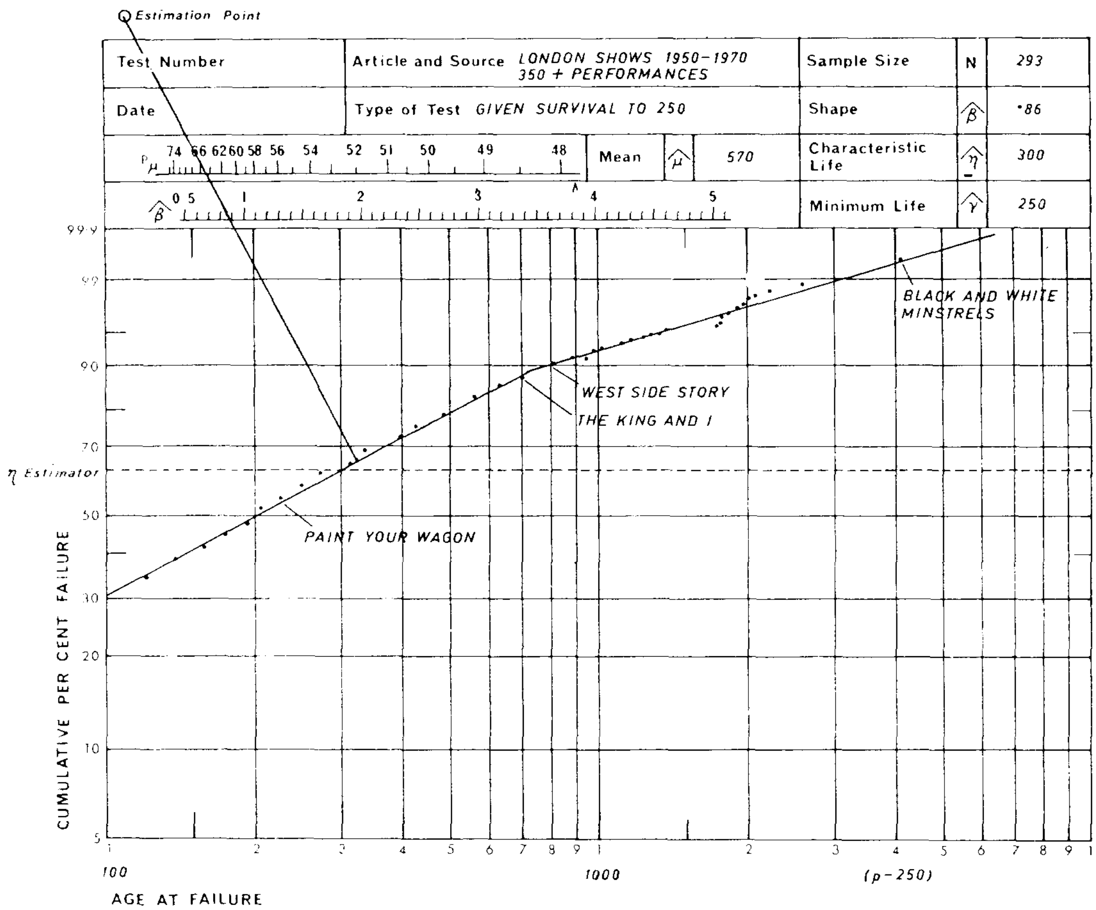

FIG. 3—WEIBULL PLOT OF LONDON SHOWS (1950-1970) WHICH SURVIVED 350 OR MORE PERFORMANCES, GIVEN SURVIVAL TO 250

Thus the probability of a show surviving p performances, given survival to 1000 performances,

$$P\ (p|1000)=\exp\left[-\frac{p-1000}{850}\right]$$

i.e. beyond 1000 performances a show is subject to random failure.

Absolute Probability of Survival

Using the data provided in *Who's Who in the Theatre* it has only been possible to calculate the conditional probability of survival, given survival to 250 performances. An investor, or 'angel' in theatrical parlance, needs rather to know the absolute probability of survival for a number of performances beyond the opening night.

By Bayes Theorem:

$$P(p)=P(p|250)\times P(250)$$

i.e. the probability of surviving a number of performances (p) equals the probability of surviving that number, given survival to 250, times the probability of surviving those first 250 performances.

This figure P(250) could be determined by dividing the total number of shows which had a first night in London theatres between 1950 and 1970 by the 293 which having opened survived 250 performances. However no record of this number could be found.

Assuming P(250)=0·4 (i.e. 40 per cent. of all shows which open survive 250 performances), then for p greater than 250 and less than 1000

$$P(p)=P(p|250)\times 0{\cdot}4$$

and for p greater than 1000

$$P(p)=P(p|1000)\times P(1000|250)\times 0{\cdot}4$$

Example:

West Side Story opened at Her Majesty's Theatre in December 1958 and ran for 1040 performances. What was the probability of this run?

$$P(250)=0{\cdot}4, \text{ say}$$

$$P(1000|250)=\exp\left[-\left(\frac{1000-250}{300}\right)^{\cdot 86}\right]=0{\cdot}111$$

$$P(1040|1000)=\exp\left[-\left(\frac{1040-1000}{850}\right)\right]=0{\cdot}954$$

Hence

$$P(1040)=0{\cdot}4\times 0{\cdot}111\times 0{\cdot}954=0{\cdot}042, \text{ i.e. } 4{\cdot}2\%$$

A backer could, of course, in view of *West Side Story*'s previous successful run in New York say 'I am certain that the show will run for 250 performances' (or any number he may wish) and estimate his chances accordingly, e.g.

$$P(1040) \text{ (given } P(250)=1)=1\times 0{\cdot}111\times 0{\cdot}954=0{\cdot}106, \text{ i.e. } 10{\cdot}6\%$$

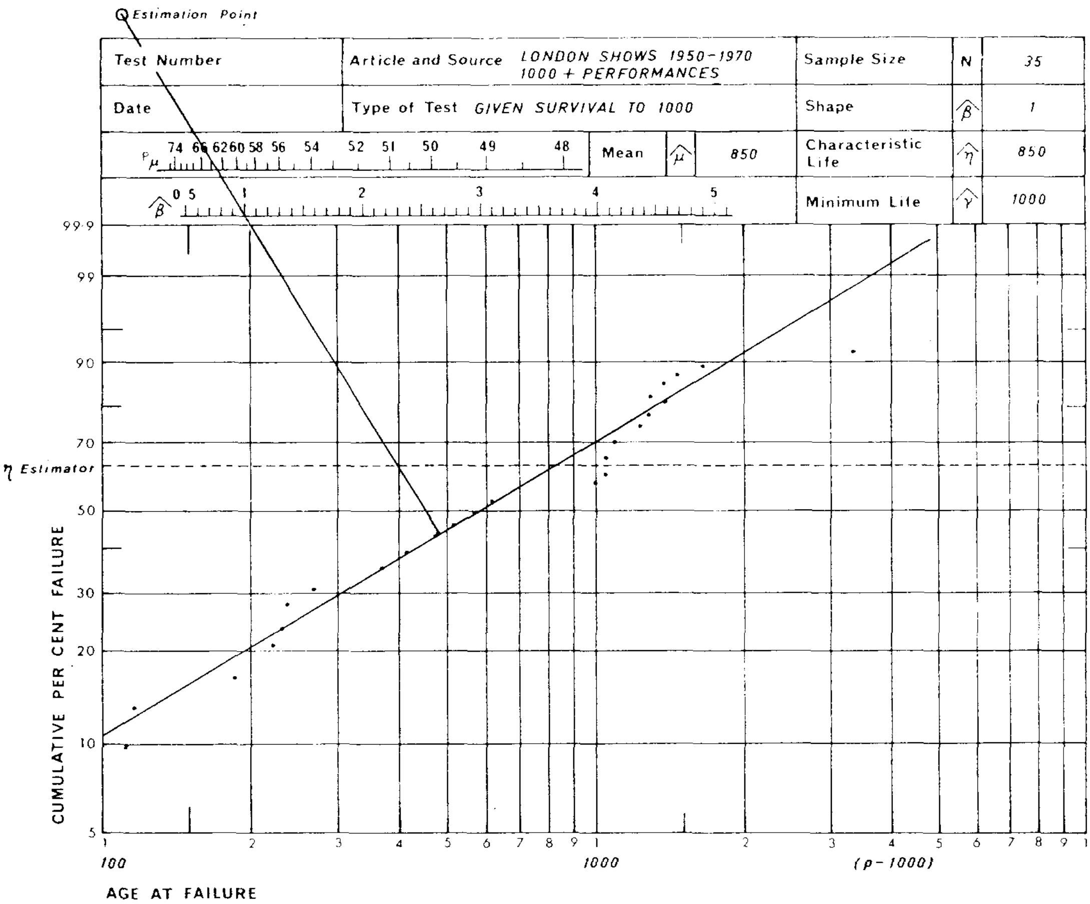

FIG. 4—WEIBULL PLOT OF LONDON SHOWS (1950-1970) WHICH SURVIVED MORE THAN 1000 PERFORMANCES, GIVEN SURVIVAL TO 1000

Comparison with Reliability in Engineering

Reliability is defined as the probability of survival which is just what has been examined in this case. In engineering, however, early life data is generally sparse. Failure data increases with age and, if a bi-modal set of data is discovered (as in Ref. 2), it tends to be due to the onset of wearout. In this case, however, we see a long 'burn in' phase followed by a period of random failure, i.e. the beginning rather than the end of the renowned 'bathtub'. It is significant to note that a change from a falling to a constant hazard rate appears on a Weibull plot, as in FIG. 3, as a reduction in slope. It is very tempting when analysing data by a Weibull plot to assume that if the plot is bi-modal the characteristics of both parts can be read directly from the combined plot. This analysis of West End Shows demonstrates just how far wrong such a technique would be; for it would yield for the latter part of the plot a shape factor of 0·5, whereas in fact beyond 1000 performances, given survival to that number, the shape factor is 1, i.e. failures are randomly distributed.

Further Analysis

It seems reasonable to postulate that the distribution of times to failure may result from a combination of:

(*a*) a random distribution of failures over the whole range, together with

(*b*) some extra distribution of failures over the early period 1 to 1000.

Such a model of the system would then be a series logic arrangement of two components D_1 and D_2, where D_1 has a random failure distribution:

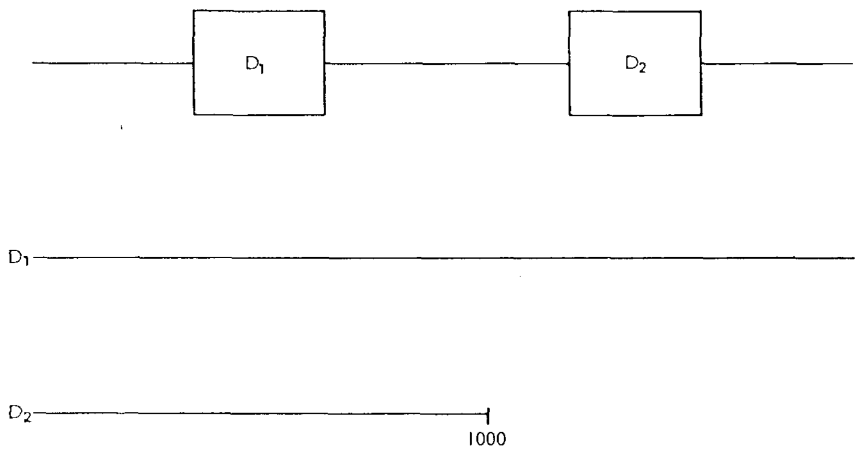

FIG. 5

Using simple reliability relationships it is possible to identify the failure distribution of component D_2 in this theoretical model.

If

R_1=reliability of D_1, and R_2=reliability of D_2

then the whole system reliability between 250 and 1000 is

$$R_S = R_1 R_2, \text{ i.e. } R_2 = \frac{R_S}{R_1}$$

Now R_1 can be identified from the distribution of failures beyond 1000 where component D_1 only is assumed to exist. In this region

$$P(p|1000) = \exp\left[-\left(\frac{p-1000}{850}\right)\right]$$

which is the familiar negative exponential expression with failure rate $\frac{1}{850}$.

Thus

$$R_1 = \exp\left[-\left(\frac{p-250}{850}\right)\right]$$

and R_s can be identified from the distribution of failures between 250 and 1000 where both D_1 and D_2 contribute to failure.

$$R_s = P(p|250) = \exp\left[-\left(\frac{p-250}{300}\right)^{\cdot 86}\right]$$

Hence

$$R_2 = \frac{R_s}{R_1} = \exp\left[-\left(\frac{p-250}{300}\right)^{\cdot 86} + \left(\frac{p-250}{850}\right)\right] \text{ for } (250 < p < 1000).$$

This equation is shown graphically in FIG. 6.

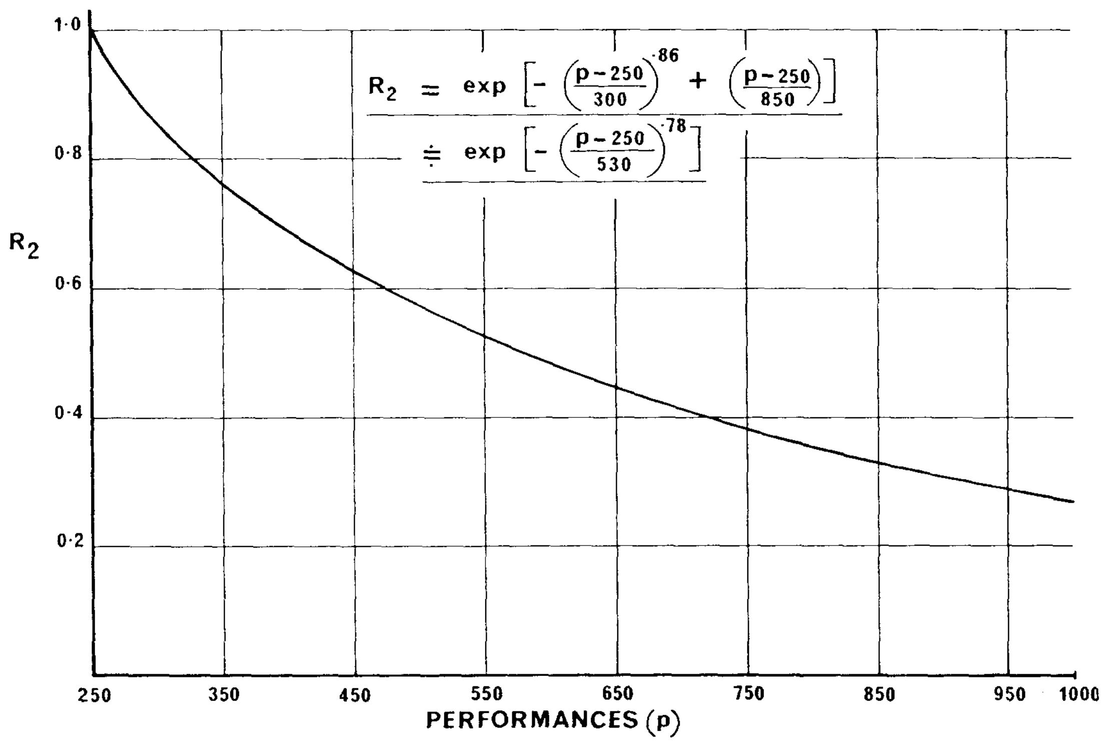

FIG. 6

Calculating R_2 for a range of values of p between 250 and 1000 and plotting F_2 $(=1-R_2)$ on Weibull probability paper yields the following very close approximation

$$R_2 = \exp\left[-\left(\frac{p-250}{530}\right)^{\cdot 78}\right] \text{ for } 250 < p < 1000$$

Examples

The King and I opened at Drury Lane in October 1953 and ran for 946 performances. The probability of this run, given survival for 250 performances, is

$$P(p|250)=\exp\left[-\left(\frac{p-250}{300}\right)^{\cdot 86}\right]$$

$$=\exp\left[-\left(\frac{946-250}{300}\right)^{\cdot 86}\right]=0{\cdot}127,\ \text{i.e. } 12{\cdot}7\%$$

i.e. the probability of failure before 946 performances is 87.3%.

Consider the proposed system model, given survival to 250 performances. The probability of avoiding random failure (the failure distribution of component D_1) is

$$R_1=\exp\left[-\left(\frac{p-250}{850}\right)\right]$$

$$=\exp\left[-\left(\frac{696}{850}\right)\right]=0{\cdot}44$$

and the probability of avoiding failure of component D_2 is

$$R_2=\exp\left[-\left(\frac{p-250}{530}\right)^{\cdot 78}\right]=0{\cdot}29$$

The combined probability of survival to 946 performances is therefore

$$R_s=R_1R_2=0{\cdot}128$$

giving

$$F_s=1-R_s=0{\cdot}872,\ \text{i.e. } 87{\cdot}2\%$$

which compares well with:

(*a*) 87·3% above

(*b*) the median rank plotting position for *The King and I*, 87·4%

(*c*) 88% from the Weibull plot, FIG. 3.

Similarly, *Paint Your Wagon* opened at Her Majesty's Theatre in February 1953 and ran for 477 performances. For this number of performances, the probability of avoiding random failure is

$$R_1=\exp\left[-\left(\frac{227}{850}\right)\right]=\cdot 766$$

and the probability of avoiding failure of component D_2 is

$$R_2=\exp\left[-\left(\frac{227}{530}\right)^{\cdot 78}\right]=\cdot 597$$

so the combined probability of survival is

$$R_s=R_1R_2=0{\cdot}457$$

and hence

$$F_s=54{\cdot}3\%$$

cf. the median rank plotting position for *Paint Your Wagon* which was 55·56% and the value given by the best fit straight line of the Weibull plot, 54·3%.

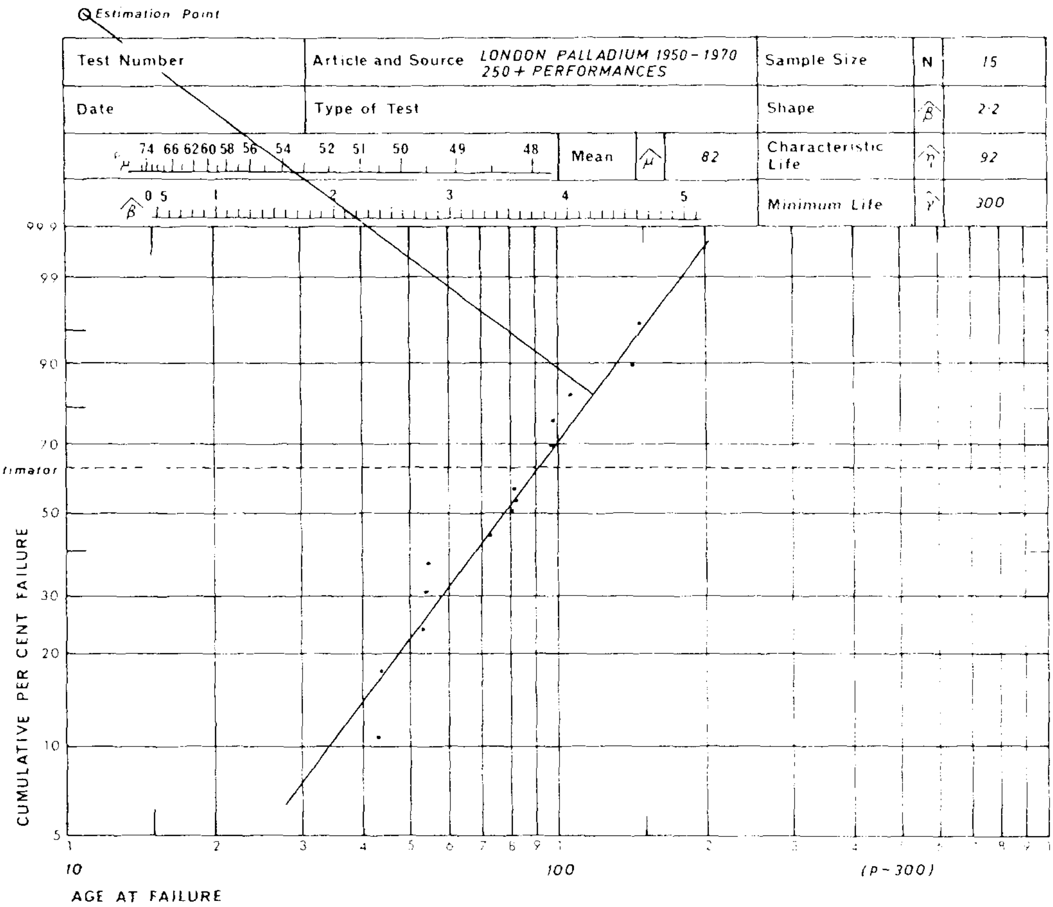

FIG. 7—WEIBULL PLOT OF LONDON PALLADIUM SHOWS (1950-1970) MORE THAN 250 PERFORMANCES

The London Palladium

Returning to the original set of data, it will be recalled that the shows at the London Palladium were excluded. This was done because the data did not appear to be consistent with the rest of the set. As can be seen from FIG. 1, shows which lasted more than 250 performances all lasted between 300 and 550. The data and common experience suggest that shows at the London Palladium, unlike those at other theatres, are not allowed to run for as long as they show a profit but are replaced at some predetermined time. A Weibull plot of the 15 shows is given at FIG. 7. With such a scatter it is perhaps not valid to use a Weibull distribution, but for what it is worth the characteristics of the best fit line are:

Shape factor $\beta=2{\cdot}2$
Characteristic life $\eta=92$
Minimum life$=300$

yielding:

$$P(p|250)=\exp\left[-\left(\frac{p-300}{92}\right)^{2.2}\right]$$

In reliability a shape factor of 2 or more, indicating an increasing hazard rate, is associated with 'wear-out'. In general, however, it is a measure of the consistency of the event. The higher the value of shape factor the greater the probability that the event will occur at or near the characteristic life.

Weibull analysis of the London Palladium Shows which include, for instance, *Doddy's Here*! (442 performances) and *Doddy's Here Again* (355 performances) is probably more informative about the management of the Palladium than about the audience appeal of the star.

Conclusion

A study that began as a fairly light-hearted search for interesting data to analyse turned out to have unexpected value in demonstrating how a bi-modal Weibull plot can be analysed. It is a pity that running hours-to-failure data for naval equipment is so much less accessible.

Perhaps this study deserves a name. Theatrotechnology?

References:

1. McClune, Captain W. J., 'Life and Repair Patterns of Shipborne Equipment', September 1970.
2. Venton, A. O. F., 'How Numerate is Terotechnology?', *Trans.I.Mar.E.*, February 1975.