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THE VIBRATIONAL PROPERTIES OF BRANCHED TORSIONAL SYSTEMS HAVING ONE, OR MORE, BRANCH POINTS

Dr. B. Dawson* and J. M. Sidwell*

This paper presents the general properties of the natural frequencies and normal elastic curves of multi-rotor, branched torsional systems having one or two branch points. The relationship between the natural frequencies of the system and the natural frequencies of the individual branches when the branch points are clamped is shown. The characteristics are illustrated by a number of synthesized examples, which have been specifically chosen to exhibit the general properties of branched systems.

1.0 INTRODUCTION

The design of marine propulsion units often necessitates the determination of the natural frequencies and normal elastic curves of branched systems which have a single branch point and a number of multi-rotor branches originating from this point.[1] When three or more of these branches are similar, or have a number of natural frequencies in common, multiple natural frequencies of the system occur, which affects the distribution of the natural frequencies and which may also cause difficulty when using certain computational routines.

In order to optimize on a specific design that will avoid dangerous resonances, a designer requires an appreciation of the vibrational properties of branched systems. It is the objective of this paper to help provide the designer with such an appreciation. The general distribution of the natural frequencies of branched systems is presented for systems having one or two branch points. It is shown that for systems with one branch point any multiple natural frequencies are equal in value to the repeated simple natural frequencies of the individual branches when the branch point is clamped. It is also shown that a branch frequency which is repeated r times will give rise to an $(r-1)$ fold multiple natural frequency of the system.

For systems with more than one branch point it is convenient to refer to the arms connecting the branch points as "links" and the arms having one end connected to a branch point and the other end free as "branches". The general distribution of the natural frequencies for systems with two branch points is presented. However, the relationship concerning the distribution of the natural frequencies is not so explicit in this case as compared to the case of single branch point systems. It is shown, however, that by considering the system as a set of subsystems with each subsystem having only one branch point, then each

* The Polytechnic of Central London

$$\left[\begin{array}{ccc|ccc|c|c|c}
k_{1A} & -k_{1A} & O & & & & & & O \\
-k_{1A} & (k_{1A}+k_{2A}) & -k_{2A} & & O & & O & O & O \\
O & -k_{2A} & (k_{2A}+k_{3A}) & & & & & & -k_{3A} \\
\hline
 & & & k_{1B} & -k_{1B} & O & & & O \\
 & O & & -k_{1B} & (k_{1B}+k_{2B}) & -k_{2B} & O & O & O \\
 & & & O & -k_{2B} & (k_{2B}+k_{3B}) & & & -k_{3B} \\
\hline
 & O & & & O & & \ddots & O & \vdots \\
\hline
 & O & & & O & & O & \ddots & \vdots \\
\hline
O & O & -k_{3A} & O & O & -k_{3B} & \cdots & \cdots & k_{3A}+k_{3B}+k_{3C}+k_{3D}
\end{array}\right]$$

subsystem may be treated in turn as a single branch point system and in this manner a more explicit relationship concerning the distribution of the natural frequencies may be obtained.

It is also shown that the number of nodes in each branch arm may be inferred from a knowledge of the distribution of the natural frequencies of the system with respect to the individual branch natural frequencies.

2.0 SINGLE BRANCH POINT SYSTEMS

The torsional natural frequencies ω_i of a multi-rotor branched system with one branch point and a number of branches as shown in Fig. 1 are given by the eigenvalues of:

$$AX = \omega^2 BX \tag{1}$$

where X is the column matrix of the angular displacements θ_i, B is a positive diagonal matrix of inertias and A is a symmetric stiffness matrix of the type shown on the previous page.

Each row sum and column sum is zero. After premultiplying both sides of equation (1) by B^{-1} then $(B^{-1}A - \omega^2 I)X = O$ where I is the unit matrix: *i.e.* $MX = O$ and M will be of the above type with each row divided by the appropriate inertia and with $-\omega^2$ added to each element on the main diagonal. The eigenvalues of M are the squares of the resonant frequencies of the whole system. The matrix A for the general case of a single branch point system with m branches may be written in the form:

$$\begin{bmatrix} A_1 & O & \dots & O & u_1 \\ O & A_2 & \dots & O & u_2 \\ \cdot & & & & \\ \cdot & & & & \\ O & O & \dots & A_m & u_m \\ u_1^T & u_2^T & \dots & u_m^T & a \end{bmatrix}$$

where each matrix A_i is of the form:

$$\begin{bmatrix} k_{1i} & -k_{1i} & O & \dots & O & O \\ -k_{1i} & (k_{1i}+k_{2i}) & -k_{2i} & \dots & O & O \\ O & -k_{2i} & (k_{2i}+k_{3i}) & \dots & O & O \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ O & O & O & \dots & -k_{(r-1)i} & (k_{(r-1)i}+k_{ri}) \end{bmatrix},$$

$$u_i = \begin{bmatrix} O \\ \cdot \\ \cdot \\ -k_{ri} \end{bmatrix}, \quad a = \sum_{i=1}^{m} k_{ri},$$

r is the number of rotors in the i^{th} branch excluding the branch point rotor, and hence k_{ri} is the branch shaft stiffness adjoining the branch point of the i^{th} branch. Matrix B of equation (1) may be expressed in the partitioned form:

$$\begin{bmatrix} B_1 & O & \dots & O & O \\ O & B_2 & \dots & O & O \\ \cdot & & & & \\ \cdot & & & & \\ O & & \dots & B_m & O \\ O & & \dots & O & b \end{bmatrix}$$

where b is the inertia of the branch point rotor.

It is proved[2] that for such systems the following relationship applies between the eigenvalues of the total system given by $AX = \lambda' BX$ and those of the subsidiary systems given by the equations $A_i x = \lambda B_i x$, namely that if the values λ_i are arranged in ascending order $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \dots \leqslant \lambda_n$, taking into account any multiplicity of eigenvalues, then the n eigenvalues $\lambda_o' \dots \lambda_{n-1}'$ of the system satisfy the relationship.

$$\lambda_o' < \lambda_1 \leqslant \lambda_1' \leqslant \lambda_2 \leqslant \lambda_2' \leqslant \dots \leqslant \lambda_{n-1} < \lambda_{n-1}' \tag{2}$$

where λ_{n-1}' has an upper limit given by $\lambda_{n-1}' < \lambda_{n-1} + a/b$ and the condition $\lambda_i < \lambda_{i+1}$ implies $\lambda_i < \lambda_i' < \lambda_{i+1}$.

2.1 *Single Branch Point Systems without any Identical Branch Frequencies (Untuned Systems)*

For this type of system $\lambda_i < \lambda_{i+1}$ for all λ and the inequality of (2) is reduced to the form:

$$\lambda_o' < \lambda_1 < \lambda_1' < \lambda_2 < \lambda_2' < \lambda_3 < \dots < \lambda_{n-1} < \lambda_{n-1}' \tag{3}$$

with the upper limit of λ_{n-1}' given by $\lambda_{n-1}' < \lambda_{n-1} + a/b$.

It may be recognized that the partitioned matrices A_1, A_2 ... along the main diagonal of A are the stiffness matrices of the individual branches with their branch points clamped. Thus, for the system in Fig. 1, if the individual branch eigenvalues are arranged in ascending order $\lambda_1, \lambda_2 \dots \lambda_{12}$ and noting that for this system $\lambda_i < \lambda_{i+1}$ then the eigenvalues of the complete system interleave the eigenvalues of the branches in the manner:

$$\lambda_o' = O, \lambda_1 < \lambda_1' < \lambda_2 < \lambda_2' \dots < \lambda_{10} < \lambda_{11}' < \lambda_{12} < \lambda_{12}'$$

$$< \lambda_{12} + \frac{(k_{3A} + k_{3B} + k_{3C} + k_{3D})}{I_B}$$

where $\lambda_o' = O$ to account for the free body mode.

To illustrate the validity of the above criterion, a branched system was synthesized, utilizing a technique developed by the authors[3], such that the branch natural frequencies were as shown in Fig. 2.

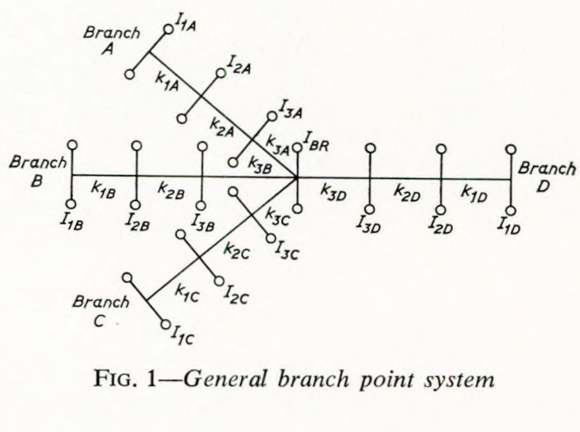

FIG. 1—*General branch point system*

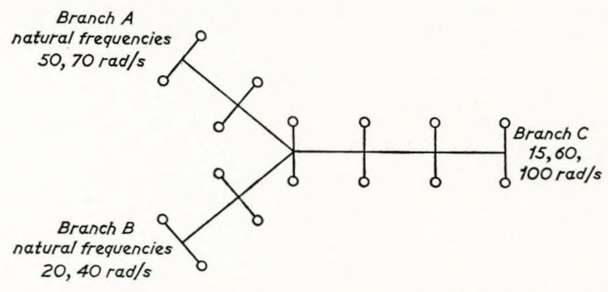

FIG. 2—*System I—No common branch frequencies*

Arranging the branch natural frequencies in ascending order and now, for convenience, letting λ represent the natural frequencies gives:

$$\lambda_1 = 15;\ \lambda_2 = 20;\ \lambda_3 = 40;\ \lambda_4 = 50;\ \lambda_5 = 60;\ \lambda_6 = 70;\ \lambda_7 = 100$$

and the natural frequencies of the completed system may be predicted according to the criterion above to be:

$$\lambda_o' = 0;\ 15 < \lambda_1' < 20 < \lambda_2' < 40 < \lambda_3' < 50 <$$

$$\lambda_4' < 60 < \lambda_5' < 70 < \lambda_6' < 100 < \lambda_7' < K$$

where $K = \sqrt{\dfrac{100^2 + 4\,472\,667 + 4\,022\,800 + 65\,104\,895}{2000}}$

$$= 216{\cdot}33$$

The natural frequencies of the system (System I) have been obtained by a matrix procedure for the whole system and are shown in Table I. It can be observed that the natural frequencies fall within the specified frequency bounds.

The eigenvector information obtained from the matrix solution is also summarized in Table I in terms of the number of nodes in each arm for the particular natural frequencies of the system. It can be seen that an extra node is introduced into a branch once a natural frequency of that branch has been passed through. This is to be predicted since at a natural frequency of a branch the branch point will be a node as the branch natural frequencies were calculated with the branch point clamped. At a higher frequency the wavelength will be less and the node will move from the branch point into the branch.

The branch natural frequencies arranged in ascending order and letting λ now represent the natural frequencies are:

$$\lambda_1 = 10;\ \lambda_2 = 15;\ \lambda_3 = \lambda_4 = \lambda_5 = 20;\ \lambda_6 = 40;\ \lambda_7 = \lambda_8 = 50;$$

$$\lambda_9 = 60;\ \lambda_{10} = \lambda_{11} = \lambda_{12} = 70;\ \lambda_{13} = 100$$

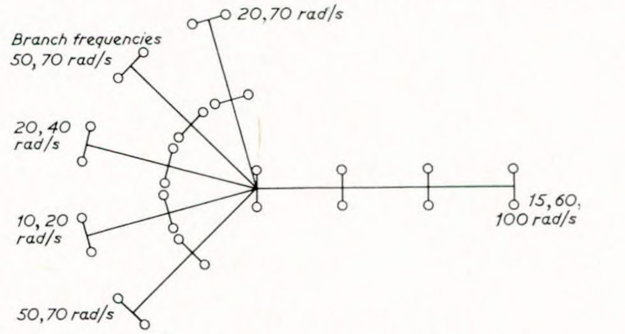

FIG. 3—*System II—Common branch frequencies*

The natural frequencies of the complete system may by reference to equation (2) be predicted to interlace the natural frequencies of the individual branches in the following manner:

$$\lambda_o' = 0;\ 10 < \lambda_1' < 15 < \lambda_2' < 20 = \lambda_3' = 20 = \lambda_4' =$$

$$20 < \lambda_5' < 40 < \lambda_6' < 50 = \lambda_7' = 50 < \lambda_8' < 60 < \lambda_9'$$

$$< 70 = \lambda_{10}' = 70 = \lambda_{11}' = 70 < \lambda_{12}' < 100 < \lambda_{13}' \leqslant 284$$

TABLE I—SYSTEM I CALCULATED RESULTS

λ' number	System natural frequency rad/s	Number of nodes		
		50, 70 rad/s branch	20, 40 rad/s branch	15, 60, 100 rad/s branch
0	0	0	0	0
1	17·19	0	0	1
2	24·08	0	1	1
3	40·14	0	2	1
4	59·93	1	2	1
5	64·71	1	2	2
6	99·996	2	2	2
7	199·4	2	2	3

2.2 *Single Branch Point Systems with Common Branch Frequencies (Tuned Systems)*

It follows from the general inequality expressed in equation (2) that when an eigenvalue of a branch is common with that of another branch, then the system zero lying between them takes their common value. Also, if a value of λ_i is repeated r times and λ_j is the next value and is repeated s times, then λ_i counts $(r - 1)$ times and λ_j $(s - 1)$ times among the system eigenvalues and there is one eigenvalue strictly between them.

A system (System II) synthesized to illustrate the above properties is shown in Fig. 3. It has three branches with natural frequencies of 70 rad/s, three branches with natural frequencies of 20 rad/s and two branches with natural frequencies of 50 rad/s.

As expected, this predicts two repeated natural frequencies at 20 rad/s and 70 rad/s with a single natural frequency at 50 rad/s. For comparison of the predicted results to actual theoretical values, the natural frequencies and normal elastic curves have been computed by a matrix method and are shown in Table II. It can be observed that the natural frequencies of the complete system fall within the predicted ranges and it may also be observed that extra nodes are introduced into the branch arms as previously predicted. The normal elastic curves of the 20, 50 and 70 rad/s natural frequencies of the complete system are of interest in that only those branches having these particular branch frequencies resonate. The remaining branches and the branch rotor remain stationary and these are sometimes referred

TABLE II—SYSTEM II CALCULATED RESULTS

λ' number	System natural frequencies	Number of nodes per branch					
		50, 70 rad/s branch	20, 40 rad/s branch	10, 20 rad/s branch	20, 70 rad/s branch	50, 70 rad/s branch	15, 60, 100 rad/s branch
0	0	0	0	0	0	0	0
1	10·401	0	0	1	0	0	0
2	16·487	0	0	1	0	0	1
3	20		0	1	0		
4	20		0	1	0		
5	23·083	0	1	2	1	0	1
6	40·071	0	2	2	1	0	1
7	50	0				0	
8	59·966	1	2	2	1	1	1
9	64·708	1	2	2	1	1	2
10	70	1			1	1	
11	70	1			1	1	
12	99·998	2	2	2	2	2	2
13	271·969	2	2	2	2	2	3

to as tuning fork nodes. The node counts as shown in Table II for tuning fork nodes were obtained treating the branch point rotor as a true zero and ignoring its sign. These nodes illustrate the concept of tuning whereby the designer can tune out particular branch motions by arranging for identical branch natural frequencies.

3.0 SYSTEMS WITH TWO BRANCH POINTS

The stiffness matrix of the two branch point system shown in Fig. 4 may easily be shown to be of the form presented on the facing page.

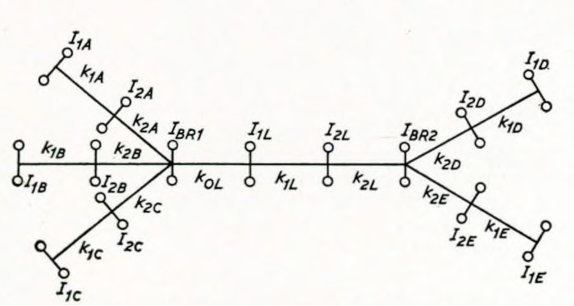

FIG. 4—*General two branch point system*

This may be expressed in a general form:

$$\begin{bmatrix}
A_{1,1} & O & \cdots & O & O & O & \cdots & O & O & \mu_1 & O \\
O & A_{2,1} & \cdots & O & O & O & \cdots & O & O & \mu_2 & O \\
\vdots & & & \vdots & & & & & & \vdots & \vdots \\
O & O & \cdots & A_{m,1} & O & O & \cdots & O & O & \mu_m & O \\
O & O & \cdots & O & A_{m+1,2} & O & \cdots & O & O & O & \upsilon_{m+1} \\
O & O & \cdots & O & O & A_{m+2,2} & \cdots & O & O & O & \upsilon_{m+2} \\
\vdots & & & & & & & \vdots & & & \vdots \\
O & O & \cdots & O & O & O & \cdots & A_{p,2} & O & O & \upsilon_p \\
O & O & \cdots & O & O & O & \cdots & O & A_{n-2} & \mu_{n-2} & \upsilon_{n-2} \\
\mu_1^T & \mu_2^T & \cdots & \mu_m^T & O & O & \cdots & O & \mu_{n-2}^T & a_{n-1} & O \\
O & O & \cdots & O & \upsilon_{m+1}^T & \upsilon_{m+1}^T & \cdots & \upsilon_p^T & \upsilon_{n-2}^T & O & a_n
\end{bmatrix}$$

$$
\left[\begin{array}{c|c|c|c|c|c|c|c}
\begin{matrix} k_{1A} & -k_{1A} \\ -k_{1A} & (k_{1A}+k_{2A}) \end{matrix} & O & O & O & O & O & \begin{matrix} O \\ -k_{2A} \end{matrix} & \begin{matrix} O \\ O \end{matrix} \\
\hline
O & \begin{matrix} k_{1B} & -k_{1B} \\ -k_{1B} & (k_{1B}+k_{2B}) \end{matrix} & O & O & O & O & \begin{matrix} O \\ -k_{2B} \end{matrix} & \begin{matrix} O \\ O \end{matrix} \\
\hline
O & O & \begin{matrix} k_{1C} & -k_{1C} \\ -k_{1C} & (k_{1C}+k_{2C}) \end{matrix} & O & O & O & \begin{matrix} O \\ -k_{2C} \end{matrix} & \begin{matrix} O \\ O \end{matrix} \\
\hline
O & O & O & \begin{matrix} k_{1D} & -k_{1D} \\ -k_{1D} & (k_{1D}+k_{2D}) \end{matrix} & O & O & \begin{matrix} O \\ O \end{matrix} & \begin{matrix} O \\ -k_{2D} \end{matrix} \\
\hline
O & O & O & O & \begin{matrix} k_{1E} & -k_{1E} \\ -k_{1E} & (k_{1E}+k_{2E}) \end{matrix} & O & \begin{matrix} O \\ O \end{matrix} & \begin{matrix} O \\ -k_{2E} \end{matrix} \\
\hline
O & O & O & O & O & \begin{matrix} (k_{0L}+k_{1L}) & -k_{1L} \\ -k_{1L} & (k_{1L}+k_{2L}) \end{matrix} & \begin{matrix} -k_{0L} \\ O \end{matrix} & \begin{matrix} O \\ -k_{2L} \end{matrix} \\
\hline
\begin{matrix} O & -k_{2A} \end{matrix} & \begin{matrix} O & -k_{2B} \end{matrix} & \begin{matrix} O & -k_{2C} \end{matrix} & \begin{matrix} O & O \end{matrix} & \begin{matrix} O & O \end{matrix} & \begin{matrix} -k_{0L} & O \end{matrix} & \begin{matrix} (\Sigma k_{2i}+k_{0L}) \\ i = A, B, C \end{matrix} & O \\
\hline
\begin{matrix} O & O \end{matrix} & \begin{matrix} O & O \end{matrix} & \begin{matrix} O & O \end{matrix} & \begin{matrix} O & -k_{2D} \end{matrix} & \begin{matrix} O & -k_{2E} \end{matrix} & \begin{matrix} O & -k_{2L} \end{matrix} & O & \begin{matrix} (\Sigma k_{2i}+k_{2L}) \\ i = D, E \end{matrix}
\end{array}\right]
$$

where the second suffix denotes reference to the branch point to which the branch arms are fixed and A_{n-2}, μ_{n-2}, υ_{n-2} are of the form:

$$A_{n-2} = \begin{bmatrix} (k_{oL}+k_{1L}) & -k_{1L} & O & \dots & O & O \\ -k_{1L} & (k_{1L}+k_{2L}) & -k_{2L} & \dots & O & O \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ O & O & O & & -k_{r-1,L} & (k_{r-1,L}+k_{rL}) \end{bmatrix};$$

$$\mu_{n-2} = \begin{bmatrix} -k_{oL} \\ \cdot \\ \cdot \\ O \end{bmatrix}; \upsilon_{n-2} = \begin{bmatrix} O \\ \cdot \\ \cdot \\ -k_{rL} \end{bmatrix}$$

r being the number of link rotors excluding the branch point rotors and the suffix L denotes reference to the link arm.

A_{i1}, A_{i2} $(i = 1 \dots p)$ and μ_i $(i = 1 \dots m)$ are of the form previously obtained for single branch point systems, υ_i $(i = m+1 \dots p)$ is of the same form as μ_i,

$$a_{n-1} = \sum_{i=1}^{N1} k_{s1_i}$$

and

$$a_n = \sum_{i=1}^{N2} k_{s2_i}$$

where $s1$ and $s2$ refer to the branch shaft sections adjoining branch points 1 and 2 respectively and $N1$, $N2$ are the number of shafts adjoining branch points 1 and 2 respectively. The inertia matrix B may also be partitioned in the form:

$$\begin{bmatrix} B_1 & O & \dots & O & O & O \\ O & B_2 & \dots & O & O & O \\ \cdot & & & & & \\ \cdot & & & & & \\ O & O & \dots & B_{n-2} & O & O \\ O & O & \dots & O & b_{n-1} & O \\ O & O & \dots & O & O & b_n \end{bmatrix}$$

where b_{n-1}, b_n are the inertias of the first branch point and the second branch point rotors respectively.

It has been proved in (2) that the following relationship applies between the eigenvalues $Ax = \lambda' Bx$ and the eigenvalues of the equations given by $A_i x = \lambda B_i x$ $(i = 1, 2 \dots n-2)$, namely:

$$\lambda_o' = 0; 0 \leqslant \lambda_1' \leqslant \lambda_2; \lambda_1 \leqslant \lambda_2' \leqslant \lambda_3;$$

$$\lambda_2 \leqslant \lambda_3' \leqslant \lambda_4; \dots \lambda_{n-4} \leqslant \lambda_{n-3}' \leqslant \lambda_{n-2};$$

$$\lambda_{n-3} \leqslant \lambda_{n-2}' \leqslant \lambda_{n-2} + \frac{a_{n-1}}{b_{n-1}};$$

$$\lambda_{n-2} \leqslant \lambda_{n-1}' \leqslant \lambda_{n-2} + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n} \quad (4)$$

It is apparent from examination of the stiffness matrix that the sub-matrices $A_{1,1} \dots A_{p,2}$ are the stiffness matrices of the individual branches considered fixed at the branch points and the sub-matrix A_{n-2} is the stiffness matrix of the link considered fixed at both ends. Thus for the system shown in Fig. 4, if the individual branch eigenvalues and link eigenvalues are arranged in ascending order $\lambda_1, \lambda_2 \dots \lambda_{12}$ then the bounds of the eigenvalues λ' of the total system will be as predicted by equation (4) above.

In order to illustrate the type of bounds given by the above criterion, the system (System III) shown in Fig. 5 was synthesized.

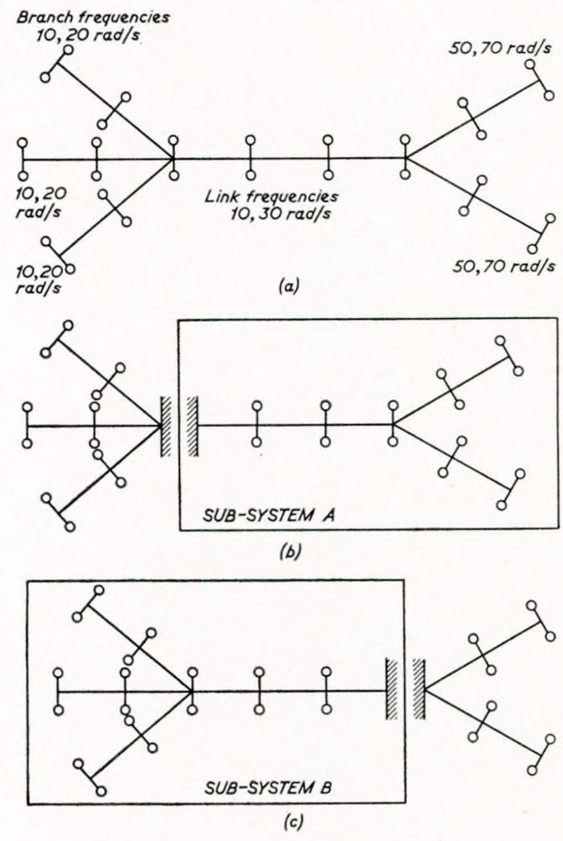

FIG. 5—*System III—Single link two branch point system*

The branch and link natural frequencies arranged in ascending order and letting λ represent tne natural frequencies are:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 10; \lambda_5 = \lambda_6 = \lambda_7 = 20;$$

$$\lambda_8 = 30; \lambda_9 = \lambda_{10} = 50; \lambda_{11} = \lambda_{12} = 70$$

The natural frequencies of the complete system may therefore be predicted by equation (4) to lie in the ranges:

$$\lambda_0' = 0;\ 0 \leqslant \lambda_1' \leqslant 10;\ 10 \leqslant \lambda_2' \leqslant 10;\ 10 \leqslant \lambda_3' \leqslant 10;$$

$$10 \leqslant \lambda_4' \leqslant 20;\ 10 \leqslant \lambda_5' \leqslant 20;\ 20 \leqslant \lambda_6' \leqslant 20;$$

$$20 \leqslant \lambda_7' \leqslant 30;\ 20 \leqslant \lambda_8' \leqslant 50;\ 30 \leqslant \lambda_9' \leqslant 50;\ 50 \leqslant \lambda_{10}' \leqslant 70;$$

$$50 \leqslant \lambda_{11}' \leqslant 70;\ 70 \leqslant \lambda_{12}' \leqslant 80{\cdot}7;\ 70 \leqslant \lambda_{13}' \leqslant 268$$

In addition, more precise information about the location of the system eigenvalues may sometimes be obtained by considering the system as a set of subsystems each with one branch point. Both branch points of the system are clamped and then released in turn, using the results of the previous section at each stage. The order in which the clamps are released will not affect the final result provided it is remembered that occasionally one of the eigenvalues of the subsystem with one branch point released will also become an eigenvalue of the whole system, with both branch points released.

A detailed discussion of when this occurs is given in reference (2). Two specific cases will be given in this paper. If a subsystem has a tuning fork mode in which the link arms are at rest, then that eigenvalue will be an eigenvalue of the final system after releasing the second branch point clamp. Alternatively, if we have two identical link arms, one of whose eigenvalues is not an eigenvalue of any of the branch arms, then these links vibrate in antiphase. The symmetry of the identical links at all frequencies assures cancellation of various terms in the system matrix and gives rise to the corresponding subsystem eigenvalue becoming an eigenvalue of the final system.

To illustrate these points we return to System III shown in Fig. 5. The natural frequencies of subsystem A will be in the ranges:

$$0 < \lambda_1' < 10;\ 10 < \lambda_2' < 30;\ 30 < \lambda_3' < 50;\ \lambda_4' = 50;$$

$$50 < \lambda_5' < 70;\ \lambda_6' = 70;\ 70 < \lambda_7' < 80{\cdot}7 \qquad (5)$$

Remembering that λ_4' and λ_6' are tuning fork modes in which the link arm is at rest, we can say that these are also natural frequencies of the whole system, with both branch points unclamped. The remaining natural frequencies of subsystem A together with those of the branches to the left of the clamped branch point can now be arranged in ascending order:

$$\lambda_1';\ 10;\ 10;\ 10;\ \lambda_2';\ 20;\ 20;\ 20;\ \lambda_3';\ \lambda_5';\ \lambda_7'$$
(assuming $10 < \lambda_2' < 20$)

Applying the results of section 2.2 we obtain:

$$\lambda_0'' = 0;\ \lambda_1' < \lambda_1'' < 10;\ \lambda_2'' = \lambda_3'' = 10;\ 10 < \lambda_4'' < \lambda_2';$$

$$\lambda_2' < \lambda_5'' < 20;\ \lambda_6'' = \lambda_7'' = 20;\ 20 < \lambda_8'' < \lambda_3';$$

$$\lambda_3' < \lambda_{10}'' < \lambda_5';\ \lambda_5' < \lambda_{12}'' < \lambda_7';\ \lambda_7' < \lambda_{13}'' < 268 \qquad (6)$$

together with:

$$\lambda_9'' = 50;\ \lambda_{11}'' = 70$$

Substituting the required ranges from equation (5) into equation (6) we see that the predicted ranges are now:

$$\lambda_0'' = 0;\ 0 < \lambda_1'' < 10;\ \lambda_2'' = \lambda_3'' = 10;\ 10 < \lambda_4'' < 30;$$

$$10 < \lambda_5'' < 20;\ \lambda_6'' = \lambda_7'' = 20;\ 20 < \lambda_8'' < 50;\ \lambda_9'' = 50;$$

$$30 < \lambda_{10}'' < 70;\ \lambda_{11}'' = 70;\ 50 < \lambda_{12}'' < 80{\cdot}7;\ 70 < \lambda_{13}'' < 268$$

If we had assumed that $20 < \lambda_2' < 30$ we would obtain the same result. Similarly, beginning with subsystem B and applying the same rules would also give the same ranges.

The natural frequencies and nodes per link and branch arms derived by a computer matrix procedure are shown in Table III and it can be seen that the calculated values lie within the predicted ranges.

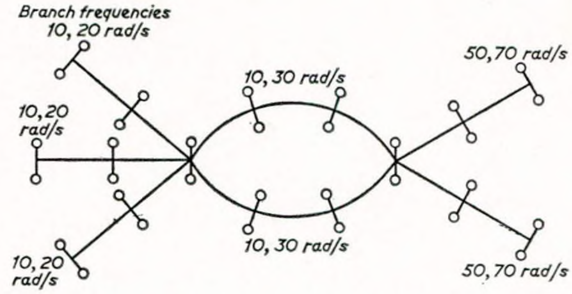

FIG. 6—*System IV—Two link two branch point system*

As a further illustrative example, a two branch point system (System IV) with two link arms, has been synthesized to have the branch and link natural frequencies shown in Fig. 6. The natural frequencies of the branch and link arms arranged in ascending order are:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 10;\ \lambda_6 = \lambda_7 = \lambda_8 = 20;$$

$$\lambda_9 = \lambda_{10} = 30;\ \lambda_{11} = \lambda_{12} = 50;\ \lambda_{13} = \lambda_{14} = 70$$

The natural frequencies of the complete system may therefore be predicted to lie in the ranges:

$$\lambda_0' = 0;\ 0 \leqslant \lambda_1' \leqslant 10;\ 10 \leqslant \lambda_2' \leqslant 10;\ 10 \leqslant \lambda_3' \leqslant 10;$$

$$10 \leqslant \lambda_4' \leqslant 10;\ 10 \leqslant \lambda_5' \leqslant 20;\ 10 \leqslant \lambda_6' \leqslant 20;$$

$$20 \leqslant \lambda_7' \leqslant 20;\ 20 \leqslant \lambda_8' \leqslant 30;\ 20 \leqslant \lambda_9' \leqslant 30;$$

$$30 \leqslant \lambda_{10}' \leqslant 50;\ 30 \leqslant \lambda_{11}' \leqslant 50;\ 50 \leqslant \lambda_{12}' \leqslant 70;$$

$$50 \leqslant \lambda_{13}' \leqslant 70;\ 70 \leqslant \lambda_{14}' \leqslant K_1;\ 70 \leqslant \lambda_{15}' \leqslant K_2$$

where K_1 and K_2 can be calculated from the stiffnesses and inertias and are found to be $K_1 = 80{\cdot}7$ and $K_2 = 268$.

Using the arguments referred to above we may improve on these inequalities and conclude that there will be three natural frequencies at 10 rad/s, two at 20 rad/s, one at 30 rad/s, at 50 rad/s and at 70 rad/s. Our improved predictions become:

$$\lambda_0' = 0;\ 0 < \lambda_1' < 10;\ \lambda_2' = \lambda_3' = \lambda_4' = 10;$$

$$10 < \lambda_5' < 20;\ 10 < \lambda_6' < 20;\ \lambda_7' = \lambda_8' = 20;$$

$$20 < \lambda_9' < 50;\ \lambda_{10}' = 30;\ 30 < \lambda_{11}' < 70;\ \lambda_{12}' = 50;$$

$$50 < \lambda_{13}' < 80{\cdot}7;\ \lambda_{14}' = 70;\ 70 < \lambda_{15}' < 268$$

The calculated natural frequencies and nodes shown in Table IV agree with these predicted ranges.

As for the case of single branch point systems, an extra node is introduced into a branch arm immediately a branch natural frequency is exceeded. Hence the next highest system natural frequency adjacent to a branch natural frequency will have an extra node in the branch arm compared to the system natural frequency below it. Any other natural frequencies of the system appearing between branch and link frequencies will introduce in turn additional nodes into the link arms. The normal elastic curves at the natural frequencies of the system caused by common branch frequencies and common link arm frequencies are of special interest. At these nodes of vibration only branches and/or links having a common frequency vibrate and the remaining branches and links remain stationary.

Table III—System III calculated results

λ′ number	System natural frequencies	Number of nodes per branch					
		10, 20 rad/s branch	10, 20 rad/s branch	10, 20 rad/s branch	50, 70 rad/s branch	50, 70 rad/s branch	10, 30 rad/s link
0	0	0	0	0	0	0	0
1	2·457	0	0	0	0	0	1
2	10	0	0	0			
3	10	0	0	0			
4	10·2	1	1	1	0	0	1
5	18·67	1	1	1	0	0	2
6	20	1	1	1			
7	20	1	1	1			
8	29·9	2	2	2	0	0	2
9	41·97	2	2	2	0	0	3
10	50				0	0	
11	64·48	2	2	2	1	1	3
12	70				1	1	
13	261·737	2	2	2	2	2	3

Table IV—System IV calculated results

λ′ number	System natural frequencies rad/s	Number of nodes per branch						
		10, 20 rad/s branch	10, 20 rad/s branch	10, 20 rad/s branch	50, 70 rad/s branch	50, 70 rad/s branch	10, 30 rad/s link	10, 30 rad/s link
0	0	0	0	0	0	0	0	0
1	3·414	0	0	0	0	0	1	1
2	10	0	0	0			0	0
3	10	0	0	0			0	0
4	10	0	0	0			0	0
5	10·42	1	1	1	0	0	1	1
6	18·741	1	1	1	0	0	2	2
7	20	1	1	1				
8	20	1	1	1				
9	29·83	2	2	2	0	0	2	2
10	30						1	1
11	43·14	2	2	2	0	0	3	3
12	50				0	0		
13	64·49	2	2	2	1	1	3	3
14	70	0			1	1		
15	262·041	2	1	1	2	2	3	3

The number of nodes per branch and link arm are shown in Tables III and IV for the systems shown in Figs. 5 and 6 respectively and it can be seen that the node counts are as predicted with the exception of the last natural frequency of the two link system. In this case examination of the normal elastic curves showed that the amplitudes of the 10, 20 rad/s branch arms were a factor of 10^{10} down on the amplitudes of the other branches and links. Rounding errors at these extremely small amplitudes (effectively zero) would account for the incorrect nodal count.

CONCLUSIONS

A comprehensive study of the vibrational properties of multi-branch, one or two branch point torsional systems has been presented, which should help to provide the designer with an appreciation of the vibrational properties of general torsional systems in terms of the natural frequencies of the individual branch and link arms. The ranges have been established within which the system natural frequencies fall and the conditions for multiple natural frequencies demonstrated. Modal analysis in terms of the nodes per branch and link arms has also been established.

The information presented should enable a designer to make effective decisions regarding optimizing designs to avoid dangerous resonant conditions.

REFERENCES

1) Salzman, R. H., 1973: "Torsional characteristics of geared turbine systems", *Marine Engineers Review, I.Mar.E.*, March.
2) Dawson, B., Goodchild, C. J. and Sidwell, J. M. "Natural Frequency Bands of Branched Vibrating Systems." PCL Dynamics Group Research Report 2.
3) Dawson, B., Muddell, C. and Sidwell, J. M. "A numerical method for the synthesis of undamped large order systems." Pro. NEL Conference, Exploitation of Vibration, Glasgow.

ACKNOWLEDGEMENTS

The authors wish to thank Mr. B. Gudgin for the computation of the eigenvalues and eigenvectors of the synthesized systems.

Discussion

Mr. D. H. L. Inns contributed to the discussion by writing that the appearance of the paper, and the erudite treatment of the subject displayed in its presentation came as a welcome surprise.

The natural frequency analysis of branched torsional systems was a specialized area within a specialized discipline and must inevitably be of limited interest to the majority of marine engineers. This, however, in no way detracts from the increasing importance of the subject, and was no excuse for the lack of attention it had received hitherto, as least in published work. The authors were accordingly to be commended for their courage in probing the mysteries of the branched system, and recognition of the value of their work was a credit to the selectivity of the Institute. It was predictable that this paper was destined to become a well-thumbed reference of the future, if only as a basis for argument of the finer points of the subject.

The branched torsional system was not a novelty, since, in its simplest form, typified by the conventional geared steam turbine marine propulsion installation, it had been studied for many years and well documented in published literature.

In its more general and complex configurations, however, it had achieved wider importance only in the last 15 years or so. This situation had resulted from the rapidly increasing adoption of branched arrangements of marine oil engine machinery. The obvious advantages of multiple medium speed oil engine arrangements geared to a single propeller, in terms of low height compactness of machinery layout, lower capital cost and flexibility of operation and maintenance, had been appreciated for larger powered vessels, particularly the special purpose fast vehicle ferries, also cruise and container ships. Similarly, branched installations had become popular in smaller vessels such as trawlers, where all the power requirements were provided by single-engine multi-purpose power units, driving through reduction gearing to the propeller and having power take-off drives for trawl winches and auxiliary power generators.

It was significant that the authors had chosen to devote this paper exclusively to the detailed study of branched system natural frequencies. In practical application, the ultimate object of torsional vibration analysis was the estimation of actual vibratory torque and stress levels in the components of the system. To this end, most recent work had tended to concentrate on developing forced-damped or frequency response techniques, which involved the discrete application of exciting torques and damping, thereby achieving the required object in a single process. This approach had been made possible and, indeed, encouraged by the availability of the high-speed digital computer to cope with the associated volume of calculation. These developments, however, appeared to neglect natural frequency analysis, presumably in the belief that such preliminary work had been rendered unnecessary by the forced-damped technique. Contrary to this view, the two processes were essentially complementary; the natural frequency analysis providing the basic understanding of the general structure of the vibration characteristics of a system and directing attention to the areas of interest, both resonant and non-resonant, requiring estimation of actual vibratory magnitudes. At Lloyd's Register of Shipping in-house computer capabilities had been developed for both natural frequency and forced-damped analysis of branched systems, and no case was investigated without an initial natural frequency survey.

By the same token, natural frequency analysis, although an essential preliminary, was insufficient in itself to establish the relative severity of vibratory conditions, even for resonances. In practical terms, the natural frequencies must be expressed as harmonic multiples of machinery speeds in order to locate the positions of potential resonant criticals in relation to operating speed ranges. The significance of such criticals, in turn, depended on the likelihood of the associated modes of vibration being excited by predictable sources of vibratory torque, and the subsequent estimation of relative severities. This aspect detracted somewhat from the claims made by the authors in their final conclusions.

The general laws defining the distribution of natural frequences of complete branched systems in relation to the individual clamped, or "noded", branch or subsystem frequencies had been clearly expressed in the paper. In this context it was gratifying to note that the authors had avoided reference to the clamped branch frequencies as "anti-resonant natural frequencies"; an unfortunate term used by some previous writers.

As implied in the paper, these points basically were markers on the frequency scale bounding the regions of possible occurrence of true natural frequencies of the system. They also coincided with the asymptotic points arising in the "residual torque" curve, when the conventional Holzer process was applied to the solution of branched systems. The clamped branch frequencies were certainly not "natural frequencies", except where there was a multiplicity of branches having identical clamped frequencies, and the corresponding modes then involved excitations and motions in those branches only. Neither were these frequencies necessarily "anti-resonant" conditions, except in cases where there was a particular application or distribution of imposed excitations. It was assumed that the authors had dwelt on the clamped branch frequencies solely as a means of presenting a clear theoretical treatment of the generalized natural frequency analysis, and they would not advocate a preliminary investigatory survey of the individual branch characteristics in practical applications, as a separate exercise, prior to evaluation of the natural frequencies of the complete system.

It was well known that if n branches, which were identical or scaled versions of one another, were joined at a common point in a system, these would be sets of multiple natural frequencies repeated $(n-1)$ times. The corresponding $(n-1)$ particular modes of vibration involved motions of these branches only and in some anti-phase configuration; were linearly independent; and should obey the rules of orthogonality. It was regrettable that the authors had not presented detailed nodal shapes for the examples treated, so that the above characteristics of these "tuning-fork" modes could be clearly demonstrated. As a further point, since these multiple frequencies and modes of systems having joined identical branches were usually recognizable and separately predictable without consideration of the remainder of the system, it was customary in practical analysis to lump together the identical branches, in order to obtain the frequencies of the rest of the modes which involve motions of all elements in the system and with the branches vibrating in-phase. This principle applied also for multiple identical links between junction points, so that, effectively, the apparent complexity of systems III and IV in the paper may be reduced to simple straight-chain systems to obtain all natural frequencies other than those of the tuning-fork variety. It was also advantageous in providing an unbroken progressive nodal count in the modes extracted; a feature which was obviously not achieved in the paper.

System II cannot be dealt with so easily, as it had the unusual feature of dissimilar joined branches variously having common clamped-mode frequencies, and therefore must be treated as a complete system. However, apart from the extreme unlikelihood of such branches occurring in practical arrangements, it also raised the question of numerical accuracy. Since the examples have been synthesized "in reverse", to produce systems having desired exact natural frequencies, it was queried whether they were actually expressed as rational systems of inertias and stiffnesses. The required mass-elastic systems would have to be quoted to an extremely high order of numerical accuracy to produce exact equality of frequencies. In practice, if it were suspected that such equalities existed in a system, it would be expedient to unbalance deliberately the branches very slightly in order to obtain identifiable separate modes differing minutely in frequency. Such unbalancing was not unrealistic, since in practical systems even "identical" branches were never precisely so, and it may be safer in the overall analysis to make this assumption.

Regarding the adoption of the matrix approach in the paper it was agreed that this resulted in an elegant and concise presentation of the theoretical aspects of branched system analysis. It was also demonstrably a viable method for practical computation. However, it should not be forgotten that there were alternative and equally effective forms of treatment. The systems selected as examples had understandably been restricted in size, consistent with satisfactorily illustrating the particular features under discussion. The most complex case, system IV, for example, has only 16 inertias. At the same time it was clear that an "all-out" solution has been carried out, evaluating all the natural frequencies of the system in each case. These two aspects of the work were not characteristic of practical torsional vibration analysis. Typical branched machinery arrangements may require analysis of systems having up to 100 inertias. Similarly, in practice, a complete solution was not necessary since it was normally sufficient to obtain frequencies and modes in a limited frequency range, determined on the basis of shaft speeds and the highest likely orders of harmonic excitation. Thus the full matrix array, typifying practical requirements, was both laborious to set up and uneconomical to process, unless some reduction procedure, taking account of the essentially tridiagonal format of the problem was resorted to. The requirement for evaluation of modes in a limited frequency range also appeared to create problems in the matrix solution technique. Scanning and iterative processes would seem to be necessary, together with infallible means for identification of modes in sequence, to ensure that none within the range had been overlooked.

The point of these remarks was that the matrix approach to torsional frequency analysis apparently suffered from as many problems as were experienced in other methods, such as the conventional Holzer process. The latter method, even in its accepted form, was an effective tool for branched system frequency analysis, particularly when means were devised for overcoming automatically the irritating appearance of asymptotic conditions in the residual torques. The computer capability at the contributor's Society for routine branch system analysis was, in fact based on the Holzer concept with many modifications, including the complete elimination of the residual torque asymptote problem. The initial adoption of the Holzer approach was largely influenced by the fact that this method fitted in so well with the other aspects of overall torsional vibration analysis and to provide continuity with pre-computer days. As a routine computing tool this facility had given satisfactory service over the past twelve years; in the very rare event of its failure, it was always possible to obtain the "failed" modes by re-arrangement of the branches.

In summary, the authors had produced a valuable contribution to the understanding of torsional vibration theory, raising many intriguing topics. Their introductory intentions had been largely fulfilled, even though they had omitted to mention "nested" branch systems and the generalized "loop" configuration—after all, these are encountered only once in a lifetime. All things considered, it would be less than charitable to suggest that their concluding claims were slightly over-optimistic.

MR. C. GRAY wrote that the increase in complexity of marine diesel propulsion units and other diesel-powered machinery installations during recent years had led to the development of improved methods for calculating torsional vibration. These had been mainly for final check calculations of a proposed arrangement of engines, dampers, couplings, gears, shafting and other components. The present paper, on the other hand, described a combined-system analysis which should enable the designer to make an improved initial selection of suitable components, and the contributor was sure designers of marine machinery especially would welcome this kind of approach.

Recent work at BICERI on computer programs for torsional vibration calculations had been described in Reference (1). The computational techniques used in these programs differ from those of the paper in a major respect, namely that they eliminated all large matrices by evaluating the overall characteristics of the individual branches separately, then combining them in the form of a connexion matrix, the size of which depended on the number of branches instead of the number of masses.

In the early stages of design calculations some lumping together of adjacent masses was acceptable, whereas for final calculations it was desirable to include all the masses separately. Could the authors' methods be extended to systems with many masses, or was there a practical limit?

The authors used the term "natural frequency of a branch" to describe the frequency at which an individual branch was in resonance with the branch point clamped. Mr. Gray believed the meaning would be more readily understood if this was called the "anti-resonant" frequency of the branch, since this term was well known in connexion with dynamic vibration absorbers.

REFERENCE

1) GRAY, C. and EDWARDS, A. J. May 1974, "Torsional vibration calculations of installations with coupled multiple engines." *Diesel Engineers and Users Association* publication No. 361.

MR. R. H. SALZMAN stated Dr. Dawson and Mr. Sidwell had skilfully presented a detailed account of the torsional vibratory characteristics of branched systems which should be of significant value to marine designers and serve as a guide in the determination of natural frequencies using hand calculations.

The authors had mentioned that for tuned systems with three or more branches, numerical difficulties may be encountered in the frequency analysis when certain digital techniques were used. The Jacobi rotation method was a procedure that the contributor had successfully employed in analyzing such systems. In order to use the Jacobi technique equation (1) must be transformed into the following symmetric form:

$$(C)(Y) = \omega^2 (Y) \text{ where}$$

$$(C) = (1/\sqrt{B})(A)(1/\sqrt{B}) \text{ and } (Y) = (\sqrt{B})(X)$$

It was the opinion of the contributor that the concept of tuning described in the paper may be somewhat obscure to the reader. For example, the analysis of system II given in section 2.2 indicated that for, the tuned modes, those branches having frequencies of 20, 50, or 70 rad/s would resonate with the remaining branches tuned-out. In practice, such vibration could only

TABLE V

Crankshaft	0	240·62	930·63	1074·65	1958·73		2869·48	3422·3	3716·59	
Camshaft	0					2379·22				3926·75
Complete System	240·60	930·08	1074·41	1843·51	1962·23	2379·67	2869·89	3422·31	3716·70	

occur if the tuned branches had stimulation torques directly applied to them. In geared-turbine systems [1] tuned vibrations rarely occur since the tuned branches usually have no sources of excitation. However, in symmetric diesel propulsion systems tuned modes of vibration may occur if the engines fire out-of-phase with one another.

The general theory in the paper had brought to Mr.Salzman's mind a useful technique employed in the frequency analysis of large symmetric systems. The procedure involved using dynamically equivalent in-line mass-elastic models to determine the tuned and untuned natural frequencies of the actual system. The tuned modes may be found from a frequency analysis of one of the tuned branches of the overall system with a "dummy" inertia placed at the branch point whose value should be sufficient to create a node at that point (about ten times individual branch inertias). The untuned resonant frequencies may be determined by analyzing an equivalent system consisting of "untuned" and "tuned" sections. The "untuned" portion was simply the untuned branch of the actual system. The "tuned" portion was composed of the following transformed mass inertias I'_j and torsional stiffnesses K'_i derived from the actual system of n tuned branches with m inertial masses I_j and $m-1$ torsional stiffnesses K_i per branch:

$$I_j' = nI_j \text{ for } j = 1, m$$
$$K_i' = nk_i \text{ for } i = 1, m-1$$

A considerable reduction in computing time may be realized if the above technique was utilized *in lieu* of performing an overall analysis of the actual system.

PROFESSOR B. DOWNS wrote, saying the authors had effectively used their synthesis technique to establish systems whose analysis made a valuable contribution to the designer's appreciation of the vibrational behaviour of complex systems. The use of matrix notation and computer library suites of eigenvalue routines should present no problems to the younger generation of engineers but suffered the disadvantages of operating in a mathematical domain which conveys little appreciation of the physical situation. The comments of Lagrange on his method of writing equations of motion based on energy concepts came to mind "The methods which I present here require neither constructions nor reasoning of geometrical or mechanical nature, but only algebraic operations proceeding after a regular and uniform plan. Those who love analysis will view with pleasure mechanics being made a branch of it and will be grateful to me for having thus extended its domain".

Simple physical reasoning would lead to some of the authors' findings and perhaps shed a little more light on the problem for the more intuitive designer, Mechanical vibration must always involve motion in which one part of a system vibrates against another part or the same system. In this motion, the balanced inertia forces or torques were transmitted through the stressing of the intervening material. In the case of a single branch point system with n branches each having a branch point clamped frequency of ω_n, the first of these n branches oscillating at frequency ω_n and amplitude A_1 may oscillate against the second branch with amplitude A_2 (counterphase) in a balanced torque situation in which the branch point was nodal and the remainder of the system was at rest. This was the first of the multiple "tuning fork" frequencies and A_1 and A_2 may both be multiplied by any scalar M. The remainder of the $(n-1)$ tuning fork frequencies may be regarded as branch 1 with amplitude A_1 vibrating against branches 3 to n with amplitudes A_3 to A_n. Combinations of these modes may exist simultaneously with arbitrary scaler multipliers.

e.g. mode $1 \times M_1$
gave amplitude $M1 \times A1$ on branch 1 and $M1 \times A2$ (c.p.) on branch 2

mode $2 \times (-M_1)$
gave amplitude $-M1 \times A1$ on branch 1 and $-M1 \times A3$

(c.p.) on branch 2 superimposed gave $M1 \times A2$ (c.p.) on branch 2 and $M1 \times A3$ on branch 3, whilst branch 1 was stationary. The vibration of branch 2 against branch 3 was therefore not independent of the $(n-1)$ modes at the repeated frequency ω_n which involved motion of branch 1.

In a two branch point system vibration of a limb attached to the L.H. branch point against a limb attached to the R.H. branch point requires torque transmission through the link and the branch points were no longer nodal so that tuning fork frequencies of this type did not occur.

Tuning fork frequencies would arise in a multi-link two branch system when one or more of the links had equal natural frequencies when both branch points were clamped. The link arms did not need to be identical, as the authors suggested. Anti-phase vibration of two links would occur with the branch points nodal and the remainder of the system stationary. Again, there would be $(n-1)$ modes for n links having the same double clamped frequencies and these modes might occur simultaneously in varying amplitude combinations as described earlier for the arms of a single branched system.

MR. J. RICE said that working for an engine builder, he welcomed the paper in which the occurrence of the natural frequencies of branched systems had been formalized. The technique was to some extent used by his company as for instance in the judging the II and III node frequencies which occurred when a tuned damper was fitted to reduce II node resonance peaks in simple engine systems. With the possibility of building multi-engined geared installations it was felt that the more formalized approach would be of considerable assistance.

The following practical example may be of interest. A recent case of governor malfunction was eventually traced to resonance in the camshaft of the branched system shown in Fig. 7. Table V shows the lower natural frequencies of the chain drive/camshaft branch, crankshaft and line shafting and the complete system.

It would have been of value both for prediction purposes and checking to have had a knowledge of the interlacing of the frequencies as presented by the authors in their paper.

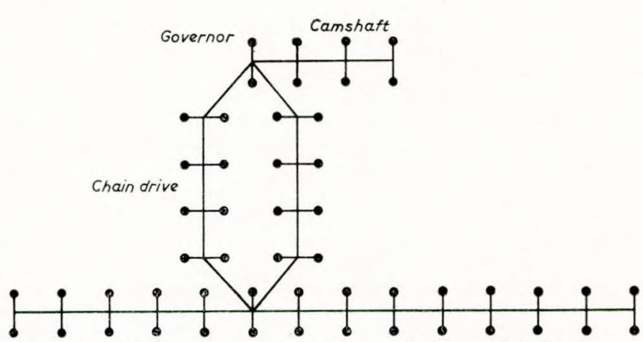

FIG. 7

Turning now to the predictions for system IV in the figure, the contributor would be pleased to have the following point clarified. Although the logic leading to the improved predictions was in order, these improved predictions seemed to be matched by an equal number of poorer predictions. Thus, referring to the two predictions, although λ'_8 is very much improved the prediction for λ'_9 had changed from $20 \leqslant \lambda'_9 \leqslant 30$ to $20 \leqslant \lambda'_9 \leqslant 50$. Were the best of both sets of predictions to be used?

Authors' reply

Dr. B. Dawson and Mr. J. Sidwell wrote in reply that they wished to thank all the contributors for their interesting comments and additional observations regarding the vibrational characteristics of branched systems.

Mr. D. H. L. Inns brought out very clearly the two aspects of vibration analysis required for a complete study of the vibrational problem namely:

i) a natural frequency analysis;
ii) a forced reponse analysis.

As so ably pointed out, the significance of resonant criticals was dependent on the likelihood of the associated modes of vibration being excited by predictable sources of vibratory torque and the relative severities. The same point was made by Mr. Salzman, who pointed out that unless a stimulation torque was present the tuned modes would not in fact resonate. Thus for a complete understanding of the torsional vibrational properties of a complex system, both a free and forced vibration analysis was desirable, and the authors acknowledged that in view of this they were a little over enthusiastic in the concluding sentence of their conclusions.

The authors agreed completely with Mr. Inns' remarks concerning the use of the term "anti-resonant" frequency. They appreciated, however, that this term had frequently been used in previous literature in connexion with branched systems, and that some engineers would recommend the use of this term. Indeed, Mr. C. Gray had made the comment that he would have preferred the authors to have used the term in the paper. Also in reply to Mr. Inns, whilst it was true that the authors had dwelt upon the determination of the clamped branch frequencies in order to present the theory of eigenvalue distribution, they also considered that it would be advantageous to work out the separate branch frequencies. This:

a) would indicate the limits for the frequencies of the complete system in terms of the separate branch frequencies and hence help to identify search ranges, assuming a search technique was being utilized;
b) it would give the designer an appreciation of how alteration of particular branch components affected the overall system frequencies.

This line of reasoning was supported by the contribution from Mr. J. Rice who considered it would have been of value both for prediction and checking purposes to have had a knowledge of the interlacing of branch frequencies when dealing with the practical example he presented of a governor malfunction.

Both Mr. Inns and Mr. R. H. Salzman had referred to the use of a dynamically equivalent model when dealing with symmetrical systems and naturally the authors recommended this method since it did reduce considerably the amount of computational time required to solve a problem. In general, however when dealing with symmetrical systems the authors recommended that as well as creating a dummy inertia at the branch point of the tuned branches, the same procedure should be followed for the "untuned" branches. The total set of tuned and untuned branch frequencies obtained would immediately indicate the bounds on all the eigenvalues of the complete system thus helping both for prediction and checking purposes.

In regard to Mr. Inns' point regarding the numerical accuracy of the synthesis procedure the Inertia and Stiffness values were synthesized and the values produced to the accuracy of the computer. This degree of accuracy was required in order to generate particular identical frequencies for the different branch systems, so that the predictions of the theory for identical eigenvalues could be illustrated.

Most of the contributors had made comments on the matrix method of solution used in the paper. Perhaps a few comments on this aspect of the paper may be pertinent although the choice of method had little relevance to the aims and objective of the paper.

The matrix approach adopted for solution of the examples had one great virtue, namely that it was extremely simple to program even for the most complex branched systems. All that was required from the user was the setting up of a stiffness and mass matrix and routine library programs may then be used for the eigenvalue and eigenvector solutions. No search problems were encountered and multiple and pathologically close eigenvalues presented no problems. Naturally like most numerical procedures numerical difficulties may be encountered for particular sets of data although these difficulties may be overcome by the use of special techniques such as the one indicated by Mr. Salzman. The authors, however, acknowledged that the matrix eigenvalue method used in the paper did have limitations, particularly:

1) in relation to the size of computer required for systems with a large number of inertias (for marine systems the number of inertias might be well over 100);
2) in respect to the determination of only a few select frequencies within a specified frequency range;
3) as expressed so eloquently by Professor B. Downs "operating in a mathematical domain which conveys little appreciation of the physical situation".

The particular choice of method of solution depended on a number of factors and each company would normally have a favourite method, the choice of which may have been dictated by previous history, or size of available computer, or indeed it may have been a purely arbitrary decision. It was however, obvious that if only a small store digital computer was available a matrix eigenvalue solution would not be a feasible method for the solution of a large order system. In this case the Holzer method or matrix transfer method as described by Mr. Gray were suitable alternative procedures.

Returning to Professor Downs' comments on physical reasoning the authors felt that he had highlighted a very important point, and found his intuitive reasoning regarding the various systems interesting and illuminating. It was agreed completely that physical reasoning should always be used in the engineering field together with a mathematical approach, since it would sometimes shed a little more light on the problem than directly apparent from the mathematics. System IV in the paper was a case that illustrated this point quite effectively. It was clear from the system matrix that if the link arms were identical, cancellation of certain terms would occur and would cause the subsystem natural frequency to become a system natural frequency ($\lambda'_{10}=30$). Professor Downs quite rightly pointed out by physical reasoning that this would also be the case when the link arms were not identical but had one natural frequency (with both branch points clamped) in common. This result was more obscure and hence more difficult to predict in the mathematical analysis.

The point referred to by Mr. Rice regarding the improved predictions (by logical reasoning) apparently leading to poorer predictions for some eigenvalues was an interesting observation. In fact the final set of predictions were an improvement on the original set, and this may be explained as follows: the first prediction for system IV gave $20 \leqslant \lambda'_8 \leqslant 30$; $20 \leqslant \lambda'_9 \leqslant 30$;

$30 \leqslant \lambda'_{10} \leqslant 50$; $30 \leqslant \lambda'_{11} \leqslant 50$. For the improved predictions the authors knew that one of λ'_8 or λ'_9 had become 20. The authors did not know which one so they called it the lowest λ'_8. In this case λ'_8 and λ'_9 originally had the same range so there was no deterioration in the range of λ'_9 so far. It was also known that one of λ'_9, λ'_{10}, λ'_{11} had become 30. The authors did not know which one, so that if $\lambda'_{10}=30$ was chosen, one of the remaining ones λ'_9, must cover the total original range of both λ'_9 and λ'_{10} i.e. $20 < \lambda'_9 < 50$. This was an improvement on the original prediction since one value was now tied down to 30 while the other covered the original total range. One cannot combine the two predictions without knowing which one of λ'_9, λ'_{10} had taken the value 30. Similarly, one of λ'_{11} and λ'_{12} would take the value 50 and again the remaining one covered the total original range of the two.

In conclusion, the authors would like to say that they felt the value of the paper had been considerably increased by the contributions, especially in relation to its value to practising marine engineers, and they would like to express their gratitude to the contributors for their interest.